HKDSE Physics and M2

BY TOBY LAM

This article was originally published on Toby Lam's blog, which hosts a variety of content on topics ranging from mathematics, technology, to life at Oxford and career advice. For more, visit `tobylam.xyz`.

A lot of the formulae[7] given to you in HKDSE Physics, as it turns out, can be derived from the calculus taught in M2. In this series of posts we're going to go through deriving some of them. For a more detailed treatise on this topic, I would highly recommend checking out the dynamics lecture notes[8] , which is a course for first year mathematics at Oxford.

We would look at rectilinear motion in part I, projectile/circular motion in part II and waves in part III.

[7] https://www.hkeaa.edu.hk/DocLibrary/HKDSE/Subject_Information/phy/Phy-Formulae-e.pdf

[8] https://courses.maths.ox.ac.uk/pluginfile.php/3628/mod_resource/content//DynamicsLectureNotes2022_updated.pdf

Part I: Rectilinear motion

Rectilinear motion is one-dimensional motion along a straight line. Due to it only having one dimension, all properties about the system could be represented by one variable only. We wouldn't need to deal with coordinates.

Consider some point particle with constant mass m. As we've seen in M2, we can respectively let displacement, velocity and acceleration as functions of time

$$\begin{aligned} \text{Displacement } &= r(t) \\ \text{Velocity } &= v(t) = \frac{dr}{dt} \\ \text{Acceleration } &= a(t) = \frac{d^2r}{dt^2}. \end{aligned}$$

Under this language, we can reframe Newton's First law as

$$\text{Momentum } = p(t) = mv(t) = m\frac{dr}{dt}$$

and Newton's second law as

$$\text{Force } = F(t) = \frac{dp}{dt} = m\frac{dv}{dt} = ma.$$

Introducing Assumptions

To get any further, we need to introduce some assumptions in DSE physics. In rectilinear motion we assume that

1. Force is constant (e.g. gravitational force)

   This means that acceleration is constant! We would now write $a(t)$ as a as it's just a constant. This is crucial as it means that

$$\begin{aligned} \frac{d^2r}{dt^2} &= a \\ \frac{dr}{dt} &= at + C_1 \\ r(t) &= \frac{1}{2}at^2 + C_1t + C_2 \end{aligned}$$

by repeated indefinite integration for some constants C_1, C_2. Naturally, we ask what those constant are. We can see that

$$\begin{aligned} v(0) &= a \cdot 0 + C_1 = C_1 \\ r(0) &= \frac{1}{2}a \cdot 0 + C_1 \cdot 0 + C_2 = C_2 \end{aligned}$$

So C_1 is the velocity at $t = 0$. C_2 is the displacement at $t = 0$, which is generally taken to be 0.

Finally putting it all together we have

$$v(t) = at + v(0)$$
$$r(t) = \frac{1}{2}at^2 + v(0)t + r(0).$$

Does this look familiar?

Conservation of Energy

To see why energy is conserved, we must first define the kinetic energy of a point particle at time t to be

$$T(t) = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2$$

and the potential energy for a point particle with displacement r (under constant force) to be

$$V(r) = -mar.$$

From DSE physics, we know that energy is conserved. I.e. $T + V$ is kept constant. However this is rather unobvious. Note how kinetic energy is with respect to time, but potential energy is with respect to displacement. In general, why would something with respect to time be related to something with respect to displacement?

It turns out that for energy to be conserved, the force needs to be conservative. In the one dimensional case, this means that there must exist a potential energy function $V(r)$ such that $F(r) = -\frac{d}{dr}V(r)$. This also means that the force is dependent on displacement only: If you are at the same displacement at different times, the force experienced is still the same.

For the case of DSE physics, as the acceleration/force is kept constant we could have $V(r) = -mar$, so the force is conservative. Note how we can add any constant to $V(r)$ and it would still be a valid potential function. Refer to the dynamics lecture notes for a more general analysis on conservative forces.

Now how do we show conservation of energy for this specific case? There's two ways of doing it. Either we expand all the terms as follows

$$\begin{aligned}(T+V) &= \left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - mar\right] \\ &= m\left[\frac{1}{2}\left(at+v(0)\right)^2 - a(\frac{1}{2}at^2 + v(0)t + r(0))\right] \\ &= m\left[\frac{1}{2}a^2t^2 + v(0)at + \frac{1}{2}v(0)^2 - \frac{1}{2}a^2t^2 - av(0)t - ar(0)\right] \\ &= \frac{1}{2}mv(0)^2 - mar(0)\end{aligned}$$

Or we can do it more abstractly by considering the derivative of $T+V$

$$\begin{aligned}\frac{d}{dt}(T+V) &= \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V(r)\right] \\ &= \frac{1}{2}m\cdot 2\frac{d^2r}{dt^2}\cdot\frac{dr}{dt} + \frac{dV}{dr}\frac{dr}{dt} \\ &= m\frac{d^2r}{dt^2}\cdot\frac{dr}{dt} - m\frac{d^2r}{dt^2}\cdot\frac{dr}{dt} \\ &= 0\end{aligned}$$

product and chain rule

$$-m\frac{d^2r}{dt^2} = -F(r) = \frac{dV}{dr}$$

So $T+V$ is constant.

In particular, this means that

$$\frac{1}{2}mv(t)^2 - mar(t) = \frac{1}{2}mv(0)^2 - mar(0).$$

Does this look familiar?

Part II

We would now look into projectile motion and uniform circular motion.

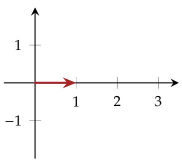
I would highly recommend checking out a video on vectors [9] before reading the post. Having a general idea of what vectors are would be extremely helpful.

[9] https://youtu.be/fNk_zzaMoSs

Motion on the 2D Plane

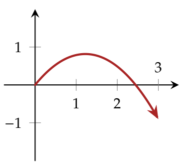
To study motion on the 2D plane, we need the idea of curves. The trajectory of a moving particle naturally forms a curve as time varies.

Mathematically, we model a curve as a function from $\mathbb{R}$, the real numbers, to $\mathbb{R}^2$, the cartesian plane. Here are some examples below.



Straight Line. The function $r(t) = (t, 0)$, for $0 < t < 1$ defines the curve shown. You could imagine it as a ball moving 1 unit on the x-axis from $t = 0$ to $t = 1$. Without doing any mathematics, you could intuitively see that the velocity is going to be constant and so acceleration would be 0.

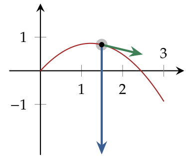
Mathematically, we can take the derivative of $r(t)$ by taking the derivative of its components. So we would have $r'(t) = (1, 0)$. This would be the velocity of the ball, a constant, unit vector pointing towards the x-axis. The acceleration, as you can guess, would be $r''(t) = (0, 0)$ which is the 0 vector.



Parabola. The function $r(t) = (3t, 4t - \frac{9.81}{2}t^2)$ for $0 < t < 1$ defines the curve shown. You could imagine it as throwing a ball at the origin under the effect of gravity. Could you try to understand this motion by considering the coordinates separately and using the equations we developed in Part I?

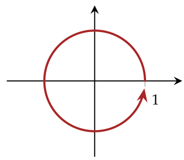
Intuitively, we know that the velocity would have the same x-component for all time t and that the acceleration would be a constant vector pointing downwards.

Mathematically, we have $r'(t) = (3, 4 - 9.81t)$ and $r''(t) = (0, -9.81)$. In the figure, the green vector is the velocity and the blue vector is the acceleration. Both vectors' magnitude are scaled down by a factor of $1/3$.



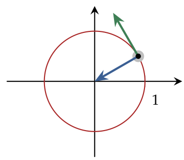
$t = 0.497$

Circle. The function $r(t) = (\cos t, \sin t)$ for $0 < t < 2\pi$ defines the curve shown. You can imagine as a ball uniformly rotating around the origin with radius 1.



Intuitively, we know that the velocity would be the tangent vector to the circle. The magnitude would be constant (1) as the motion is uniform. Acceleration would also be constant and pointing towards the origin.

Mathematically, we have $r'(t) = (-\sin t, \cos t)$ and $r''(t) = (-\cos t, -\sin t)$, which aligns with our intuition. Once again the green vector is the velocity and the blue vector is the acceleration.



$t = \pi/6$

As you can see the amount of behaviour we can model with curves (the explicit construction of the $r(t)$ function is called curve parameterization) is highly unconstrained! It is powerful enough to describe a far wider range of curves than just plots of $y = f(x)$ (which one can imagine as plotting $r(t) = (t, f(t))\ \forall t \in \mathbb{R}$). There are other ways of constructing curves such as using level sets.

Projectile Motion

Similar to part I, the crucial assumption in DSE projectile motion is that the only force exerted on the particle is the gravitational force. So once again we have

$$\begin{aligned} r''(t) &= (0, -g) \\ r'(t) &= (C^{\cdot}_1, -gt + C_1) \\ r(t) &= (C^{\cdot}_1 t + C^{\cdot}_2, -\frac{1}{2}gt^2 + C_1 t + C_2) \end{aligned}$$

for some constants $C^{\cdot}_1, C^{\cdot}_2, C_1, C_2$ by repeated "integration". Similar to part I we could find those constants in terms of initial velocities/displacement. As such most properties of projectile motion could be analysed by splitting into x and y-axis.

Another way of looking at it courtesy of Henry Yip [10] would be to consider

[10] https://henry-yip.github.io/

$$r(t) = (C^{\cdot}_2, C_2) + (C^{\cdot}_1, C_1)t + (0, -g/2)t^2$$

which tells you that for small t, $r(t)$ looks like a straight line starting from initial displacement $(C^{\cdot}_2, C_2)$ with the direction of initial velocity $(C^{\cdot}_1, C_1)$. Gradually the quadratic term dominates and we get the parabolic shape. This idea is similar to Taylor expansions.

Perhaps, then, the most interesting aspect about projectile motion is the conservation of energy. Why is it that energy is still conserved when we use the magnitude of the velocity vector in kinetic energy (instead of one dimensional velocity)? How does the formalism developed in part I relate to the 2 dimensional case? Let's make some definitions first.

Let $r(t) = (x(t), y(t))$. So $x(t), y(t)$ are the x and y components of $r(t)$ respectively. As such we have $r'(t) = (x'(t), y'(t))$ and $r''(t) = (x''(t), y''(t))$. Now we have

$$\begin{aligned} \text{Kinetic energy } &= T = \frac{1}{2}m(x'(t)^2 + y'(t)^2) \\ \text{Potential energy } &= V = mgy(t) \end{aligned}$$

So we have

$$
\begin{aligned}
T + V &= \frac{1}{2}m(x'(t)^2 + y'(t)^2) + mgy(t) \\
&= \frac{1}{2}m(C_1^{\cdot\,2} + (-gt + C_1)^2) + mg(-\frac{1}{2}gt^2 + C_1 t + C_2) \\
&= m\left[\frac{1}{2}C_1^{\cdot\,2} + \frac{1}{2}g^2t^2 - gtC_1 + \frac{1}{2}C_1^2 - \frac{1}{2}g^2t^2 + gC_1t + gC_2\right] \\
&= \frac{1}{2}m(C_1^{\cdot\,2} + C_1^2) + mgC_2
\end{aligned}
$$

which is the total energy at initial time.

A more proper way of doing this would involve multivariable calculus. Again refer to the dynamics lecture notes for a more general analysis on conservative forces.

Uniform Circular Motion

Let's think about a ball uniformly rotating around the origin. We know that two variables completely determine its behaviour, its radius and its velocity. As such we can parameterize $r(t) = (R\cos(kt), R\sin(kt))$ where R is the radius and k is some variable that as it turns out is related to angular velocity.

To intuitively see why k is related to angular velocity: Consider how $r(t) = (\cos(t), \sin(t)), 0 < t < 2\pi$ is one full anticlockwise rotation around the unit circle, but $r(t) = (\cos(2t), \sin(2t)), 0 < t < \pi$ is the same full anticlockwise rotation in half the time. We doubled k and the time taken is halved. Could you guess a relationship between k and angular velocity before we do the maths?

Let's find out the velocity and the acceleration. We have

$$\begin{aligned} r(t) &= (R\cos(kt), R\sin(kt)) \\ r'(t) &= (-Rk\sin(kt), Rk\cos(kt)) \\ r''(t) &= (-Rk^2\cos(kt), -Rk^2\sin(kt)) \end{aligned}$$

These formulae immediately tell us all we know about uniform circular motion!

Firstly, $r(t) \perp r'(t) \perp r''(t)$ from simple coordinate geometry (or you could use the dot product if you are familiar with linear algebra).

Secondly, the magnitude of the velocity is

$$\sqrt{R^2k^2(\sin^2(kt) + \cos^2(kt))} = Rk.$$

So we now know $v = Rk$.

What about angular velocity? We see that for a full anticlockwise rotation to take place, t needs to go from 0 to $2\pi/k$. The total angular change would be 2π. As such the angular velocity is $\frac{2\pi k}{2\pi} = k$. So k is the angular velocity!

Finally, $r''(t) = -k^2 r(t)$, so $a = k^2 R$!

As such we also have $a = v^2/R$

Part III: Waves

In the final part, we would discuss waves. How do we formulate waves mathematically? Why are waves often depicted by sine curves?

Wave Equation

The wave equation is the (partial differential) equation that describes all sorts of waves (water, sound, light ...) It can be written compactly as

$$\ddot{u} = c^2 \nabla^2 u.$$

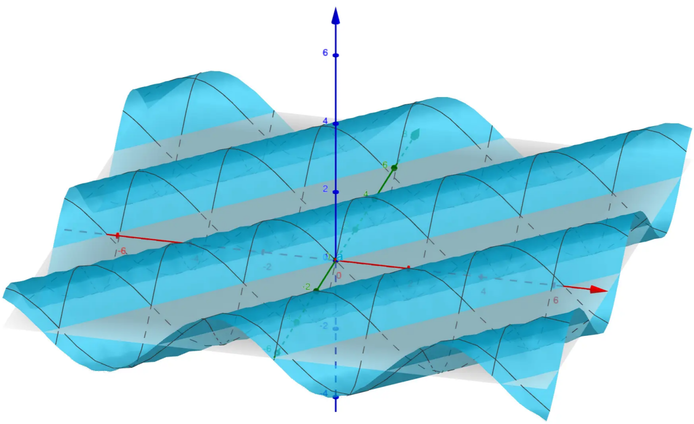
Unfortunately, to understand and derive the above would involve heavy calculus, even if we confine ourselves to one-dimensional waves.

Instead, we would like to explore the mathematical formulation of the sinusoidal travelling wave. The sinusoidal travelling wave is one of many solutions to the wave equation and is the one studied extensively in DSE physics.

The one-dimensional sinusoidal travelling wave could be represented by $u(t, x) = A \sin(kx - wt + \psi)$ where x is distance and t is time for some constants A, k, w, ψ. Try guessing what physical meaning those constants have! It would be revealed at the end.

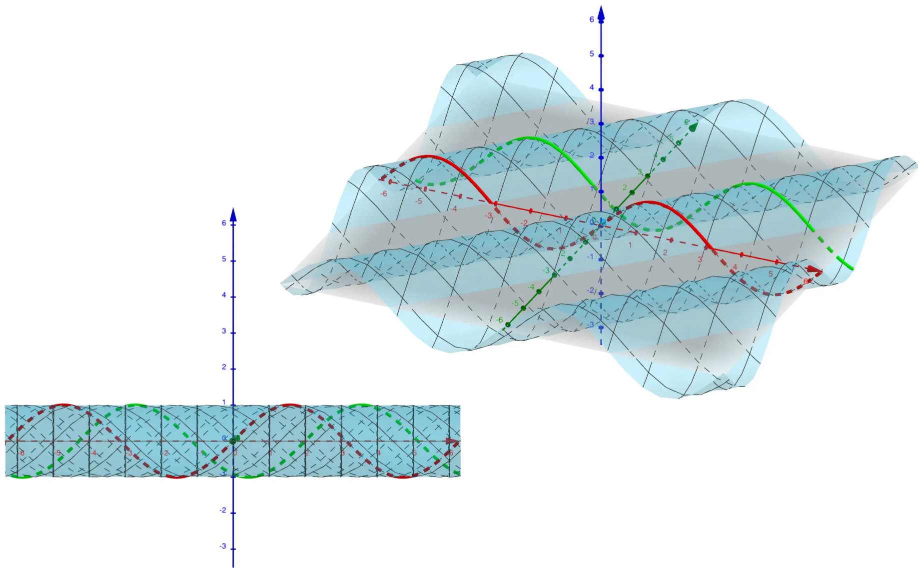
You could imagine this as a function from $\mathbb{R}^2$ to $\mathbb{R}$. It takes in time and distance and tells you the displacement of the wave.

In the graph below, we took $A = w = k = 1$ and $\psi = 0$. The x-axis is red and the t-axis is green.



We can look at how the wave looks like at time c by considering the intersection of the graph $u(t, x)$ and the plane $t = c$.

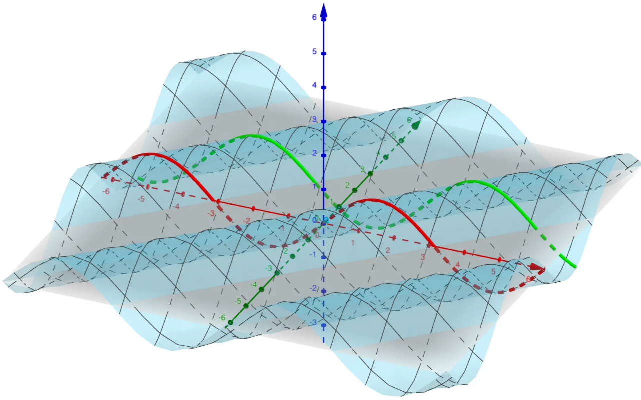
In the graph below, we took $t = 0$ for the red curve and $t = 2$ for the green curve.

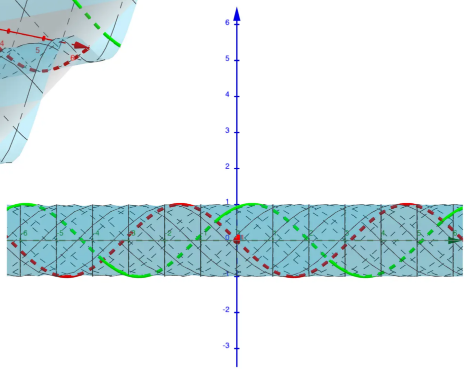


If we look at the graph along the time axis, does this look like displacement time graphs? Can you guess what direction the wave is travelling to? How could we change the direction of the wave? What is the wavelength? How does the wavelength correspond to the constants?

Similarly, if we're interested at a particular distance $x = c$, we could look at the intersection of the graph $u(t, x)$ and the plane $x = c$.

In the following graph, we took $x = 0$ for the red curve and $x = 2$ for the green curve.





If we look at the graph along the distance axis, does this look like displacement distance graphs? What is the period of the wave? How does the period correspond to the constants?

Answers and More Questions

Turns out, the wavelength λ is equal to $1/k$ and w is the (angular) frequency of the wave. Can you deduce why that is the case?

What about ψ? What does it represent?

Can you think of how to parametrize stationary waves using a similar $u(x,t)$?

Further Reading

For more on one-dimensional wave equations, there is a LibreTexts article[11] which explains more.

[11] https://chem.libretexts.org/Courses/Pacific_Union_College/Quantum_Chemistry/2%3A_The_Classical_Wave_Equation/.01%3A_The_One-Dimensional_Wave_Equation